Classical Mechanics

Lagrangian and Hamiltonian Dynamics

Introduction:

This content presents Lagrangian and Hamiltonian dynamics, an advanced formalism for studying various problems in mechanics. Lagrangian techniques can provide a much cleaner way of solving some physical systems than Newtonian mechanics, in particular the inclusion of constraints on the motion. Lagrangian techniques allow postulation of Hamilton's Principle of Least Action, which can be considered an alternative to Newton's second law as the basis of mechanics. The Hamiltonian formalism is introduced, which is useful for proving various important formal theorems in mechanics and, historically, was the starting point for quantum mechanics.

1.1 Degrees of Freedom, Constraints, and Generalized Coordinates

Degrees of Freedom

Obviously, a system of M point particles that are unconstrained in any way has 3M degrees of freedom.
There is freedom, of course, in how we specify the degrees of freedom;
(i) Choice of origin
(ii) Coordinate system: Cartesian, cylindrical, spherical, etc.
(iii) Center-of-mass vs. individual particles: $\{\overrightarrow{r_k}\}$ or $\{\vec{R}\ , \overrightarrow{s_k} = \overrightarrow{s_k} - \vec{R}\}$
But, the number of degrees of freedom is always the same; e.g., in the center-of-mass system, the constraint $\sum_k \overrightarrow{s_k} = 0$ applies, ensuring that $\{\overrightarrow{r_k}\}$ and $\{\vec{R},\overrightarrow{s_k}\}$ have same number of degrees of freedom.
The motion of such a system is completely specified by knowing the dependence of the available degrees of freedom on time.

Example 1:
In the elliptical wire example, there are a *priori* 3 degrees of freedom, the 3 spatial coordinates of the point particle. The constraints reduce this to one degree of freedom, as no motion in y is allowed and the motions in z and x are related. The loss of the y degree of freedom is easily accounted for in our Cartesian coordinate system; effectively, a 2D Cartesian system in x and z will suffice. But the relation between x and z is a constraint that cannot be trivially accommodated by dropping another Cartesian coordinate.

Example 2:
In the Atwood's machine example, there are a *priori* 2 degrees of freedom; the z coordinates of the two blocks. (We ignore the x and y degrees of freedom because the problem is inherently 1-

dimensional.) The inextensible rope connecting the two masses reduces this to one degree of freedom because, when one mass moves by a certain amount, the other one must move by the opposite amount.

Constraints

Constraints may reduce the number of degrees of freedom; e.g., particle moving on a table, rigid body, etc.
Holonomic constraints are those that can be expressed in the form

$$f(\vec{r_1}, \vec{r_2}, \ldots, t) = 0$$

For example, restricting a point particle to move on the surface of a table is the holonomic constraint $z - z_0 = 0$ where z_0 is a constant. A rigid body satisfies the holonomic set of constraints

$$|\vec{r_i} - \vec{r_j}| - c_{ij} = 0 \qquad \text{(i)}$$

where c_{ij} is a set of constants satisfying $c_{ij} = c_{ji} > 0$ for all particle pairs i, j.

There are three kinds of nonholonomic constraints:
(i) Nonintegrable or history-dependent constraints. These are constraints that are not fully defined until the full solution of the equations of motion is known. Equivalently, they are certain types of constraints involving velocities.
(ii) Inequality constraints; e.g., particles required to stay inside a box, particle sitting on a sphere but allowed to roll off.
(iii) Problems involving frictional forces.
Holonomic constraints may be divided into rheonomic ("running law") and scleronomic ("rigid law") depending on whether time appears explicitly in the constraints:
rheonomic: $f(\{\vec{r_k}\}, t) = 0$
scleronomic: $f(\{\vec{r_k}\}) = 0$

1.2 Virtual Displacement, Virtual Work, and Generalized Forces

Virtual Displacement

We define a virtual displacement $\{\delta\vec{r_i}\}$ as an infinitesimal displacement of the system coordinates $\{\vec{r_i}\}$ that satisfies the following criteria:
(i) The displacement satisfies the constraint equations, but may make use of any remaining unconstrained degrees of freedom.
(ii) The time is held fixed during the displacement.
(iii) The generalized velocities $\{\dot{q_k}\}$ are held fixed during the displacement.

A virtual displacement can be represented in terms of position coordinates or generalized coordinates. The advantage of generalized coordinates, of course, is that they automatically respect the constraints. An arbitrary set of displacements $\{\delta q_k\}$ can qualify as a virtual displacement if conditions (ii) and (iii) are additionally applied, but an arbitrary set of displacements $\{\delta\vec{r_i}\}$ may or may not qualify as a virtual displacement depending on whether the displacements obey the constraints. All three conditions will become clearer in the examples. Explicitly, the relation between infinitesimal displacements of generalized coordinates and virtual displacements of the position coordinates is

$$\delta\vec{r_i} = \sum_k \frac{\partial \vec{r_i}}{\partial q_k}\, \delta q_k \qquad \text{(ii)}$$

This expression has content: there are fewer $\{q_k\}$ than $\{\vec{r_i}\}$, so the fact that $\delta\vec{r_i}$ can be expressed only in terms of the $\{q_k\}$ reflects the fact that the virtual displacement respects the constraints. One can put in any values of the $\{\delta q_k\}$ and obtain a virtual displacement, but not every possible set of $\{\delta\vec{r_i}\}$ can be written in the above way.

Virtual Work

We can define virtual work as the work that would be done on the system by the forces acting on the system as the system undergoes the virtual displacement $\{\delta\vec{r_i}\}$:

$$\delta W = \sum_{ij} \overrightarrow{F_{ij}} \cdot \delta\vec{r_i} \qquad \text{(iii)}$$

where $\overrightarrow{F_{ij}}$ is the *j*th force acting on the coordinate of the *i*th particle $\vec{r_i}$.

Generalized Force

Our discussion of generalized coordinates essentially was an effort to make use of the constraints to eliminate the degrees of freedom in our system that have no dynamics. Similarly, the constraint forces, once they have been taken account of by transforming to the generalized coordinates, would seem to be irrelevant.

1.3 d'Alembert's Principle and the Generalized Equation of Motion

d'Alembert's Principle

Definition of virtual work is

$$\delta W = \sum_{ij} \overrightarrow{F_{ij}} \cdot \delta\vec{r_i} \qquad \text{(iv)}$$

where the sum includes all (constraint and non-constraint) forces. Assuming position coordinates are in an inertial frame (but not necessarily our generalized coordinates), Newton's second law tells us $\sum_j \overrightarrow{F_{ij}} = \dot{\vec{p_i}}$: the sum of all the forces acting on a particle give the rate of change of its momentum. Then we can rewrite δW:

$$\delta W = \sum_i \sum_j \overrightarrow{F_{ij}} \cdot \delta\vec{r_i} = \sum_i \dot{p_i} \cdot \delta\vec{r_i} \qquad \text{(v)}$$

But, we found earlier that we could write the virtual work as a sum over only non-constraint forces,

$$\delta W = \sum_{ij} \overrightarrow{F_{ij}}^{(nc)} \cdot \delta\vec{r_i} \qquad \text{(vi)}$$

Thus, we can derive the relation,

$$\sum_i \left[\sum_j \overrightarrow{F_{ij}}^{(nc)} - \dot{p_i}\right] \cdot \delta\vec{r_i} = 0 \qquad \text{(vii)}$$

The above equation is referred to as d'Alembert's principle. Its content is that the rate of change of momentum is determined only by the non-constraint forces. In this form, it is not much use, but the conclusion that the rate of change of momentum is determined only by non-constraint forces is an important physical statement.

We can use d'Alembert's principle to relate generalized forces to the rate of change of the momenta:

$$\sum_k F_k \delta q_k = \delta W = \sum_k \dot{p_\iota} . \delta r_i = \sum_{i,k} \dot{p_\iota} . \frac{\partial r_i}{\partial q_k} \delta q_k \quad \text{(viii)}$$

Here, $\{\delta \vec{r_\iota}\}$ and $\{\delta q_k\}$ are mutually independent. Therefore, we may conclude that equality holds for each term of the sum separately (Equation vii), providing a different version of d'Alembert's principle:

$$\sum_{ij} \overrightarrow{F_{\iota J}}^{(nc)} . \frac{\partial r_i}{\partial q_k} = F_k = \sum_i \dot{p_\iota} . \frac{\partial r_i}{\partial q_k} \quad \text{(ix)}$$

This is now a very important statement: the generalized force for the k^th^ generalized coordinate, which can be calculated from the non-constraint forces only, is related to a particular weighted sum of momentum time derivatives (the weights being the partial derivatives of the position coordinates with respect to the generalized coordinates). Effectively, we have an analogue of Newton's second law, but including only the non-constraint forces.

Generalized Equation of Motion

Now we perform the manipulation needed to make d'Alembert's principle useful. We know from Newtonian mechanics that work is related to kinetic energy, so it is natural to expect the virtual work due to a differential displacement $\{\delta \vec{r_\iota}\}$ to be related to some sort of small change in kinetic energy. We first begin with a formal definition of kinetic energy:

$$\mathrm{T} = \sum_i \frac{1}{2} m_\iota . \dot{r_i} . r_i = \mathrm{T}\ (\{q_k\}, \{\dot{q_k}\}, \mathrm{t})$$

T can be obtained by first writing T in terms of position velocities $\{\dot{r_\iota}\}$ and then using the definition of the position coordinates in terms of generalized coordinates to rewrite T as a function of the generalized coordinates and velocities. T may depend on all the generalized coordinates and velocities and on time because the $\{r_i\}$ depend on the generalized coordinates and time and a time derivative is being taken, which may introduce dependence on the generalized velocities (via the chain rule, as seen earlier). The partial derivatives of T are

$$\frac{\partial T}{\partial q_k} = \sum_i m_i . \dot{r_\iota} . \frac{\partial \dot{r_i}}{\partial q_k} = \sum_i p_i . \frac{\partial \dot{r_i}}{\partial q_k}$$

$$\frac{\partial T}{\partial \dot{q_k}} = \sum_i m_i . \dot{r_\iota} . \frac{\partial \dot{r_i}}{\partial \dot{q_k}} = \sum_i p_i . \frac{\partial r_i}{\partial q_k}$$

where in the last step we have made use of dot cancellation because the constraints are assumed to be holonomic.

Now, we have $\vec{p_\iota}$ floating around but we need $\dot{p_\iota}$. The natural thing is then to take a time derivative. We do this to $\frac{\partial T}{\partial \dot{q_k}}$ (instead of $\frac{\partial T}{\partial q_k}$) because we want to avoid second order time derivatives if we are to obtain something algebraically similar to the right side of d'Alembert's principle. We find

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) = \sum_i \dot{p_\iota} . \frac{\partial r_i}{\partial q_k} + \sum_i p_i . \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)$$

Or, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) = \sum_i \dot{p_\iota} . \frac{\partial r_i}{\partial q_k} + \frac{\partial T}{\partial q_k}$

So, $\sum_i \dot{p_\iota} . \frac{\partial r_i}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{\partial T}{\partial q_k}$ (x)

Recalling d'Alembert's principle (Equation ix), we may rewrite the above:

$$\sum_i \overrightarrow{F_\iota}^{(nc)} . \frac{\partial r_i}{\partial q_k} = F_k = = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{\partial T}{\partial q_k} \quad \text{(xi)}$$

This is the generalized equation of motion. The left side is completely determined by the non-constraint forces and the constraint equations. The right side is just derivatives of the kinetic energy with respect to the generalized coordinates and velocities. Thus, we obtain a differential equation for the motion in the generalized coordinates.

1.4 The Lagrangian and the Euler-Lagrange Equations

For conservative non-constraint forces, we can obtain a slightly more compact form of the generalized equation of motion, known as the Euler-Lagrange equations.

Generalized Conservative Forces

Now let us specialize to non-constraint forces which are conservative; i.e.,

$\vec{F}_i^{(nc)} = -\vec{\nabla}_i U(\{\vec{r}_i\})$

where $\vec{\nabla}_i$ indicates the gradient with respect to $\vec{r}_i$. Whether the constraint forces are conservative is irrelevant; we will only explicitly need the potential for the non-constraint forces. U is assumed to be a function only of the coordinate positions; there is no explicit dependence on time or on velocities, $\partial U/\partial t = 0$ and $\partial U/\partial \dot{r}_i = 0$.

Let us use this expression of generalized force

$$F_k = \sum_i \vec{F}_i^{(nc)} . \frac{\partial r_i}{\partial q_k} = -\vec{\nabla}_i U(\{\vec{r}_i\}) . \frac{\partial r_i}{\partial q_k} = -\frac{\partial}{\partial q_k} U(\{\vec{q}_i\}, t) \qquad \text{(xii)}$$

In the last step we make use of the holonomic constraints to rewrite U as a function of the $\{q_l\}$ and possibly t and realize that the previous line is just the partial derivative of U with respect to q_k.

Thus, rather than determining the equation of motion by calculating the generalized force from the non-constraint forces and the coordinate transformation relations, we can rewrite the potential energy as a function of the generalized coordinates and calculate the generalized force by gradients thereof.

The Euler-Lagrange Equations

We may rewrite the generalized equation of motion using the above relation between generalized force and gradient of the potential energy as

$$-\frac{\partial U}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \qquad \text{(xiii)}$$

Define the Lagrangian

$$L = T - U$$

Since we have assumed holonomic constraints, we have that $\partial U/\partial \dot{q}_k = 0$. This lets us replace d/dt $(\partial T/\partial \dot{q}_k)$ with d/dt $(\partial L/\partial \dot{q}_k)$, giving

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{(xiv)}$$

This is the Euler-Lagrange equation.

1.5 The Hamiltonian

We seek a conserved quantity, one whose total time derivative vanishes. We can construct one from the Lagrangian; it is called the Hamiltonian and has the form

$$H = \sum_k \dot{q}_k . \frac{\partial L}{\partial q_k} - L \qquad \text{(xv)}$$

The total time derivative of the Hamiltonian is

$$\frac{d}{dt}H = \sum_k \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \sum_k \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{d}{dt}L$$

We can use the Euler-Lagrange equation to rewrite the middle term, which gives

$$\frac{d}{dt}H = \sum_k \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \sum_k \dot{q}_k \frac{\partial L}{\partial q_k} - \frac{d}{dt}L$$

The first two terms are most of the total derivative of L; one is left only with

$$\frac{d}{dt}H = -\frac{\partial L}{\partial t} \qquad \text{(xvi)}$$

Thus, if time does not explicitly appear in the Lagrangian, the Hamiltonian is completely conserved. If the constraints are scleronomic (the potential is implicitly assumed to be conservative because we are able to calculate a Lagrangian), then time will not appear explicitly and H will definitely be conserved. It can be shown that H is the total energy of the system, H = T + U. For rheonomic constraints, H may still be conserved, but may not be the energy. We will investigate these conservation laws in more detail later.

1.6 Cyclic Coordinates and Canonical Momenta

If the Lagrangian contains $\dot{q}_k$ but not q_k, we can easily see from the Euler-Lagrange equation that the behavior of that coordinate is trivial:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

Which implies; $\left(\frac{\partial L}{\partial \dot{q}_k}\right) = p_k$

is constant or conserved and is termed the canonical momentum conjugate to q_k or the canonically conjugate momentum. The coordinate q_k is termed ignorable or cyclic. Once the value of p_k is specified by initial conditions, it does not change. In simple cases, the canonical momentum is simply a constant times the corresponding generalized velocity, indicating that the velocity in that coordinate is fixed and the coordinate evolves linearly in time.

1.7 The Principle of Least Action and the Euler-Lagrange Equation

The Euler-Lagrange equations can be derived using d'Alembert's principle. The form of the Euler equation suggests that if we take F = L, we will recover the Euler-Lagrange equations. Thus, we define the action,

$$S = \int_{t_0}^{t_1} dt\, L(q(t), \dot{q}(t), t)$$

and we require that the physical path satisfy the Principle of Least Action, that

$$\delta S = 0$$

for the physical path. This is also known as Hamilton's Principle because he was the first one to suggest it as a general physical principle. Plugging in for F = L in the Euler equation gives us the Euler-Lagrange equation,

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \qquad \text{(xvii)}$$